Layered wheels

9^{th} Slovenian International Conference on Graph Theory
Bled, Slovenia, June 23 - June 29, 2019

Dewi Sintiari, joint work with Nicolas Trotignon

ENS DE LYON

Introduction

Even-hole-free graphs (EHF graphs)

SOME TERMINOLOGY

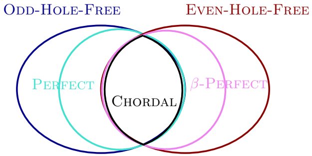
- $\mathcal{C}$ HEREDITARY if $\mathcal{C}$ is closed under taking induced subgraphs
- G CONTAINS H if H is isomorphic to an **induced** subgraph of G
- G is H-FREE if it does not contain H
- G is $\mathcal{F}$-FREE if it is H-free, $\forall H \in \mathcal{F}$

- HOLE = induced cycle of length ≥ 4
- EVEN HOLE = hole of even length
- G is EVEN-HOLE-FREE if G contains **no even hole**

Introduction

Motivation of the study of EHF graphs

- initially related to the attempt of proving Strong Perfect Graph Conjecture
- it is structurally similar to PERFECT graphs (G is perfect if $\chi(H) = \omega(H)$, for any H induced subgraph of G)
  - SPGT : G is perfect if and only if G is Berge graph (= no odd hole + no odd antihole)
  - even-hole-free = no even hole + no antihole of length ≥ 6
- its relation to β-PERFECT graphs (introduced by Markossian, Gasparian, Reed, '96)
  - G is β-perfect if $\chi(H) = \beta(H)$, for any induced subgraph H,
    where $\beta(G) = \max\{\delta(H) + 1 \; ; \; H \subseteq_{\mathrm{ind}} G\}$ and $\delta(G)$ = min-degree of vertices in G
  - trivial observation: $\chi(G) \leq \beta(G)$



Introduction

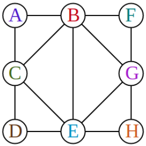
More about EHF graphs

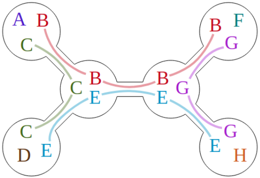
- Decomposition thm and recognition algorithm for EHF graphs are known [Conforti, Cornuéjols, Kapoor, and Vušković, 2002]
- Many graph problems are open (ex. computing χ, α)
- What to do?
  - What to study? What to exclude?
  - Bounding parameters? for ex. tree-width, clique-width, ...

Remark. For more about EHF graphs: see the survey of Kristina Vušković.

Graph parameters

Tree decomposition & Tree-width





figures taken from https://commons.wikimedia.org/wiki/User:David_Eppstein/Gallery

- The tree-width of G is a parameter measuring how far is a graph G from a tree

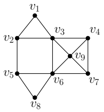
A *tree decomposition* of G is a tree $\mathcal{T}$, whose each tree node is a bag $B \subseteq V(G)$ s.t.

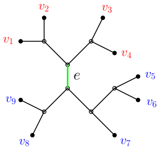
1. every vertex of G is in some bag
2. every edge is contained in at least one bag
3. for every vertex $v \in V(G)$, the set of bags containing v is connected in $\mathcal{T}$

- The *width* $\mathcal{T}$ is the size of the largest bag minus 1
- The *tree-width* of G is the width of the optimal tree decomposition

Graph parameters

Rank decomposition & Rank-width





$$\mathrm{width}(e) = \mathrm{rank}\ \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_5 & 0 & 1 & 0 & 0 \\ v_6 & 0 & 0 & 1 & 0 \\ v_7 & 0 & 0 & 0 & 1 \\ v_8 & 0 & 0 & 0 & 0 \\ v_9 & 0 & 0 & 1 & 1 \end{array} = 3$$

Hlineny et. al. Width parameters beyond tree-width and their applications, The Computer Journal (51), 2008

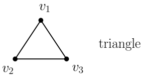
- The rank-width of G is a parameter measuring the connectivity of G

Rank decomposition is a cubic tree $\mathcal{T}$, with a bijection $\nu : V(G) \to \mathcal{L}(\mathcal{T})$

- *width*(e) : cut-rank of the adjacency matrix of the separation
- *width*($\mathcal{T}$) : $\max\{\mathrm{width}(e) \mid e \in E(\mathcal{T})\}$
- *rank-width* of G is the width of the optimal rank decomposition

Motivation

TRIANGLE-FREE EHF GRAPHS

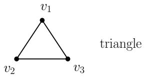


Theorem [**?**]

Every (even hole, triangle)-free graph has tree-width at most 5

Motivation

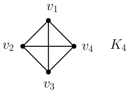
TRIANGLE-FREE EHF GRAPHS



Theorem [**?**]

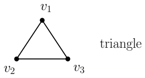
Every (even hole, triangle)-free graph has tree-width at most 5

- How about K_4-free, EHF graphs?
  (asked by Cameron, Chaplick, Hoàng)



Motivation

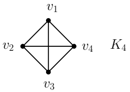
TRIANGLE-FREE EHF GRAPHS



Theorem [**?**]

Every (even hole, triangle)-free graph has tree-width at most 5

- How about K_4-free, EHF graphs?
  (asked by Cameron, Chaplick, Hoàng)

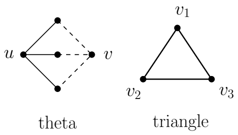


✓ No; layered wheel is a counter example

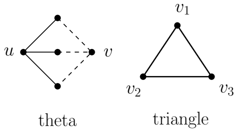
PART 1

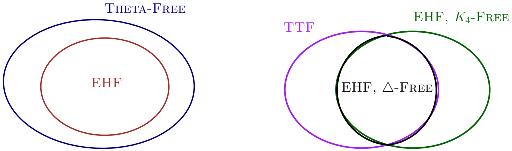
Layered wheel: construction and properties

(Theta, triangle)-free graphs (TTF)

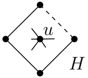


(Theta, triangle)-free graphs (TTF)

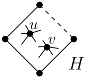




TTF graphs and EHF K_4-free graphs

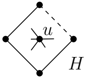


wheel

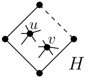


2-wheel

TTF graphs and EHF K_4-free graphs



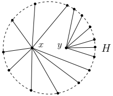
wheel



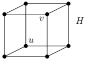
2-wheel

Structure of 2-wheels with **non-adjacent** centers:

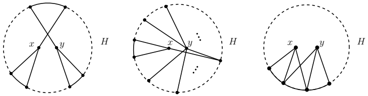
- In EHF, TRIANGLE-FREE : always nested
- In TTF : nested, except the cube
- In EHF, K_4-FREE : nested, with several exceptions



nested wheel



cube



several exceptions

Layered wheel $G_{\ell,k}$, $\ell \geq 1, k \geq 4$

(THETA, TRIANGLE)-FREE LAYERED WHEEL

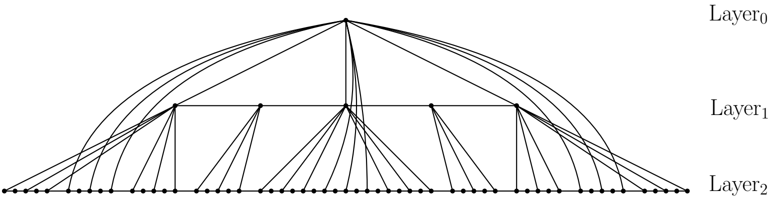
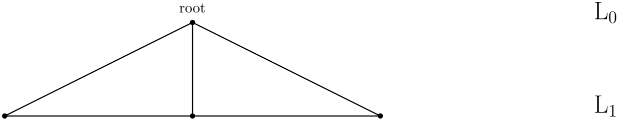


Figure: TTF Layered wheel $G_{2,4}$

TTF layered wheel construction

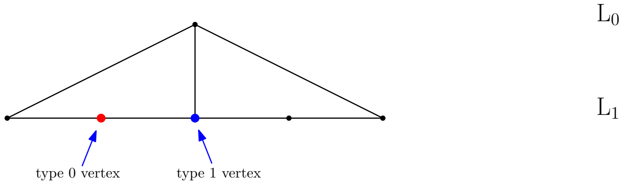
$G(\ell, k)$, with $\ell = 2$ and $k = 4$

TTF layered wheel construction



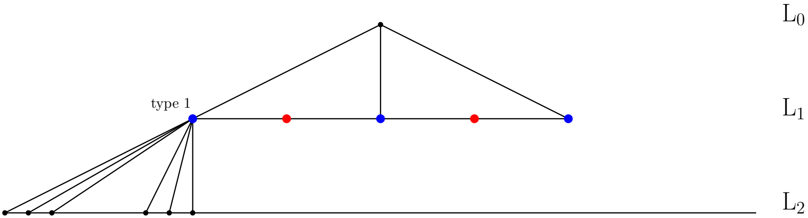
$G(\ell, k)$, with $\ell = 2$ and $k = 4$

TTF layered wheel construction



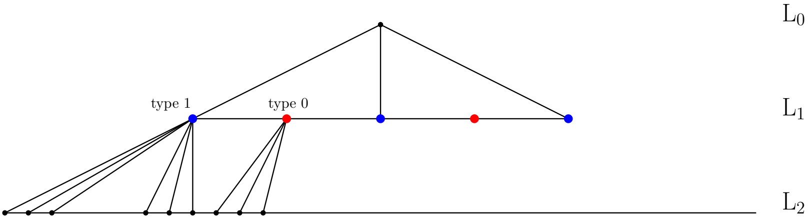
$G(\ell, k)$, with $\ell = 2$ and $k = 4$

TTF layered wheel construction



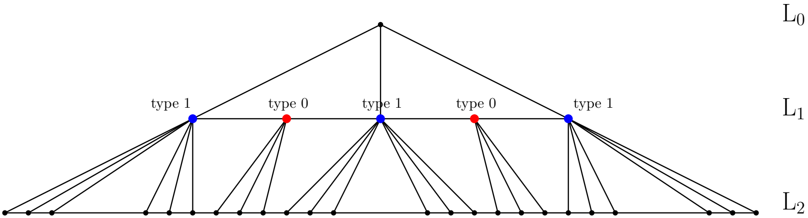
$G(\ell, k)$, with $\ell = 2$ and $k = 4$

TTF layered wheel construction



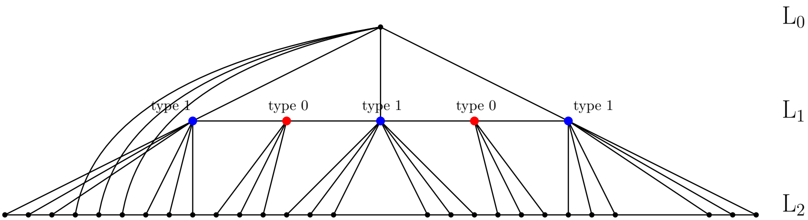
$G(\ell, k)$, with $\ell = 2$ and $k = 4$

TTF layered wheel construction



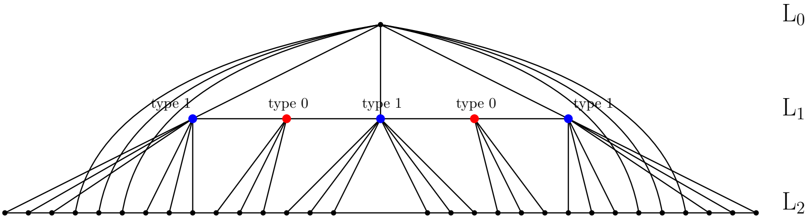
$G(\ell, k)$, with $\ell = 2$ and $k = 4$

TTF layered wheel construction



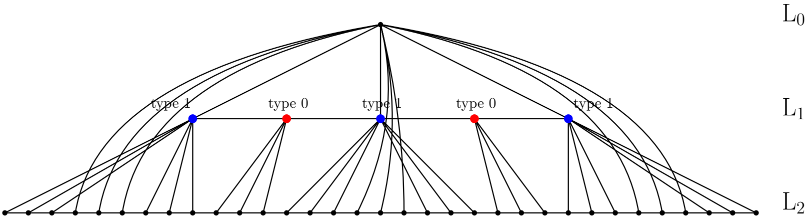
$G(\ell, k)$, with $\ell = 2$ and $k = 4$

TTF layered wheel construction



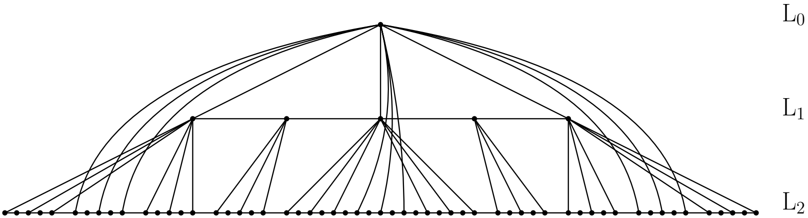
$G(\ell, k)$, with $\ell = 2$ and $k = 4$

TTF layered wheel construction



$G(\ell, k)$, with $\ell = 2$ and $k = 4$

TTF layered wheel construction



$G(\ell, k)$, with $\ell = 2$ and $k = 4$

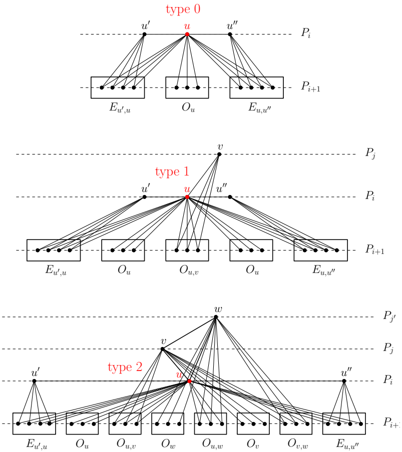
EHF layered wheel construction

Remark. For $\ell \geq 2$, $k \geq 5$, K_4-free, EHF layered wheel $G_{\ell,k}$ contains triangle.

EHF layered wheel construction

Remark. For $\ell \geq 2$, $k \geq 5$, K_4-free, EHF layered wheel $G_{\ell,k}$ contains triangle.

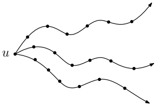
- The first two layers are similar as for TTF layered wheel
- Three types of vertices in $G_{\ell,k}$:



Layered wheel contains no theta

Remark

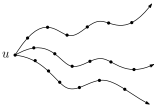
- *$G_{\ell,k}$ is full of spiders*



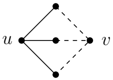
Layered wheel contains no theta

Remark

- *$G_{\ell,k}$ is full of spiders*

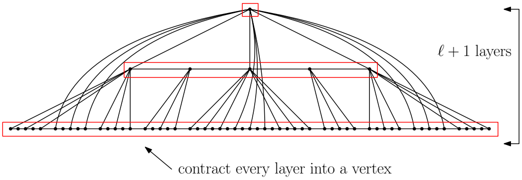


- *but... it contains no theta*



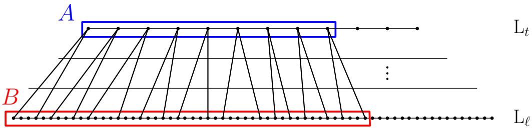
Tree-width of layered wheel

- the **tree-width** of $G_{\ell,k}$ is at least ℓ



Rank-width of layered wheel

- the **rank-width** of $G_{\ell,k}$ is at least ℓ



PART 2

What to do?

Bounding the tree-width

- The tree-width of TTF graphs and EHF, K_4-free graphs are unbounded
- Important remark: to reach tree-width ℓ, our construction needs at more then 3^ℓ vertices.

Bounding the tree-width

- The tree-width of TTF graphs and EHF, K_4-free graphs are unbounded
- Important remark: to reach tree-width ℓ, our construction needs at more then 3^ℓ vertices.

Lemma 1. $tw(G_{\ell,k}) = O(\log(|V(G_{\ell,k})|))$

Proof.

1. $|V(G_{\ell,k})| \gg 3^\ell$
2. $tw(G_{\ell,k}) \leq pw(G_{\ell,k}) \leq 2\ell$

Proof of Lemma 1

1. $|V(G_{\ell,k})| \gg 3^{\ell}$
   - Every vertex in layer L_i has at least 3^{j-i} neighbors in layer L_j

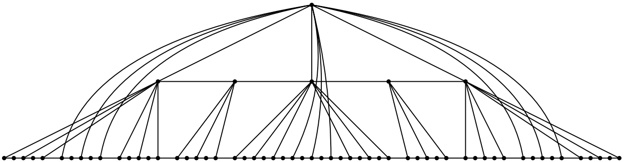


Figure: TTF layered wheel $G_{2,4}$

Proof of Lemma 1

2. $tw(G_{\ell,k}) \leq 2\ell$
   - We prove a stronger result: the *path-width* of layered wheel is at most 2ℓ.
   - $tw(G_{\ell,k}) \leq pw(G_{\ell,k}) \leq \omega(\mathcal{I}) - 1$, where $\mathcal{I}$ is an interval graph containing $G_{\ell,k}$ as a subgraph

Proof of Lemma 1

2. $tw(G_{\ell,k}) \leq 2\ell$
   - We prove a stronger result: the *path-width* of layered wheel is at most 2ℓ.
   - $tw(G_{\ell,k}) \leq pw(G_{\ell,k}) \leq \omega(\mathcal{I}) - 1$, where $\mathcal{I}$ is an interval graph containing $G_{\ell,k}$ as a subgraph

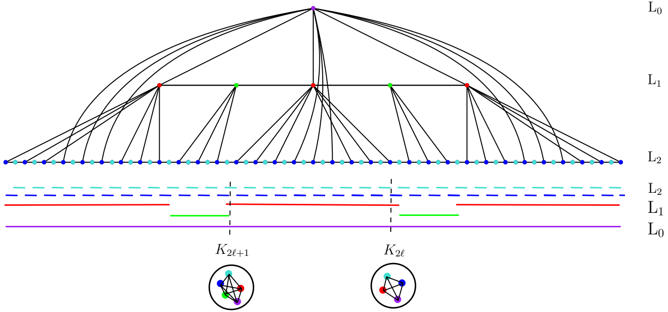


Figure: Interval graph $\mathcal{I}$ that contains $G_{2,4}$

To sum up...

Theorem 1. For every integers $\ell \geq 1$ and $k \geq 3$ there exists a graph $G_{\ell,k}$ such that:

- it is theta-free and it has girth at least k (so, is triangle-free when $k \geq 4$).
- $\ell \leq rw(G_{\ell,k}) \leq tw(G_{\ell,k}) \leq pw(G_{\ell,k}) \leq 2\ell \leq 2^{\ell} \leq |V(G_{\ell,k})|$.

To sum up...

Theorem 1. For every integers $\ell \geq 1$ and $k \geq 3$ there exists a graph $G_{\ell,k}$ such that:

- it is theta-free and it has girth at least k (so, is triangle-free when $k \geq 4$).
- $\ell \leq rw(G_{\ell,k}) \leq tw(G_{\ell,k}) \leq pw(G_{\ell,k}) \leq 2\ell \leq 2^{\ell} \leq |V(G_{\ell,k})|$.

Theorem 2. For every integers $\ell \geq 1$ and $k \geq 3$ there exists a graph $G_{\ell,k}$ such that:

- it is (even hole, K_4)-free and every hole in the graph has length at least k.
- $\ell \leq rw(G_{\ell,k}) \leq tw(G_{\ell,k}) \leq pw(G_{\ell,k}) \leq 2\ell \leq 2^{\ell} \leq |V(G_{\ell,k})|$.

Bounding the tree-width

Conjecture 1. $\exists c$ constant such that for any TTF graph G, we have

$$tw(G) \leq c \log |V(G)|.$$

Conjecture 2. $\exists c$ constant such that for any K_4-free EHF graph G, we have

$$tw(G) \leq c \log |V(G)|.$$

Bounding the tree-width

Conjecture 1. $\exists c$ constant such that for any TTF graph G, we have

$$tw(G) \leq c \log |V(G)|.$$

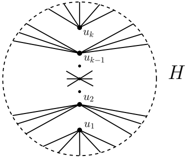
Conjecture 2. $\exists c$ constant such that for any K_4-free EHF graph G, we have

$$tw(G) \leq c \log |V(G)|.$$

✓ If so, many graph problems are poly-time solvable.

Bounding the tree-width

Partial result



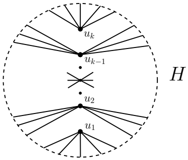
span wheel of order k

SPAN-WHEEL-NUMBER $\zeta(G)$:

the order of the largest span wheel in G

Bounding the tree-width

Partial result



span wheel of order k

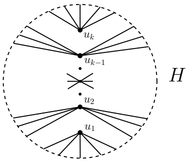
SPAN-WHEEL-NUMBER $\zeta(G)$:

the order of the largest span wheel in G

Theorem 1. Any (theta, triangle)-free graph has tree-width $O\left(\zeta(G)^{o(1)}\right)$.

Bounding the tree-width

Partial result



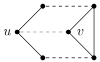
span wheel of order k

SPAN-WHEEL-NUMBER $\zeta(G)$:

the order of the largest span wheel in G

Theorem 1. Any (theta, triangle)-free graph has tree-width $O\left(\zeta(G)^{o(1)}\right)$.

Theorem 2. Any (even-hole, K_4, *pyramid*)-free graph has tree-width $O(\zeta(G)^9)$.



pyramid

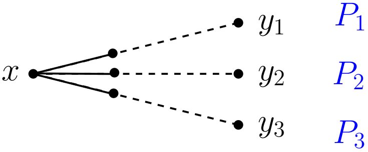
Remark. For any graph G, we have $\zeta(G) \leq \frac{n-2}{2}$

Proof sketch of Theorem 1 and Theorem 2

- Consider G TTF or K_4-free EHF
- If G has huge tree-width then either
  1. it contains a big clique
  2. it has a minimal separator of large size
- 1 is not the case (we are in triangle-free or K_4-free)
- 2 cannot be the case (because if it was then G would contain a span wheel of large order)

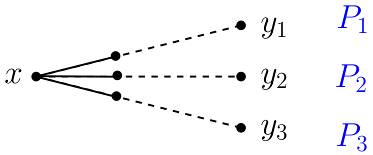
A consequence

An *m*-SPIDER, $m \geq 1$ is a graph consists of three internally-vertex-disjoint chordless paths P_1, P_2, P_3, each of length m



A consequence

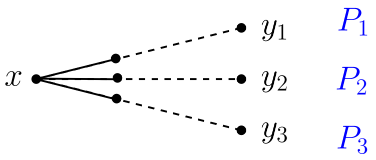
An *m*-SPIDER, $m \geq 1$ is a graph consists of three internally-vertex-disjoint chordless paths P_1, P_2, P_3, each of length m



Lemma. Let G be a triangle-free graph. Any *span-wheel* in G of at least $\lfloor \frac{3m}{2} \rfloor$ centers contains an m-spider.

A consequence

An *m*-SPIDER, $m \geq 1$ is a graph consists of three internally-vertex-disjoint chordless paths P_1, P_2, P_3, each of length m

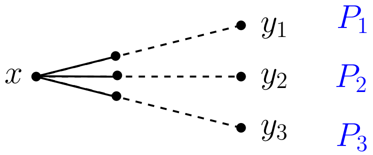


Lemma. Let G be a triangle-free graph. Any *span-wheel* in G of at least $\lfloor \frac{3m}{2} \rfloor$ centers contains an m-spider.

Theorem 3. Let $m \geq 1$. There exists a constant c such that any (theta, triangle, m-spider)-free graph G has tree-width $O\left(m^{o(1)}\right)$.

A consequence

An *m*-SPIDER, $m \geq 1$ is a graph consists of three internally-vertex-disjoint chordless paths P_1, P_2, P_3, each of length m



Lemma. Let G be a triangle-free graph. Any *span-wheel* in G of at least $\lfloor \frac{3m}{2} \rfloor$ centers contains an m-spider.

Theorem 3. Let $m \geq 1$. There exists a constant c such that any (theta, triangle, m-spider)-free graph G has tree-width $O\left(m^{o(1)}\right)$.

Remark:

- Theorem 3 is best possible in some sense
- It is conjectured that: α is poly-time computable in **spider-free** graphs
- it is conjectured that: α is poly-time computable in **theta-free** graphs

Conjecture

If G has huge tree-width, then G must contain as an induced subgraph:

- a big clique
- a big complete bipartite graph
- a big wall, possibly subdivided
- a big line graph of a subdivided wall
- layered wheels or variation of them

Conjecture

If G has huge tree-width, then G must contain as an induced subgraph:

- a big clique
- a big complete bipartite graph
- a big wall, possibly subdivided
- a big line graph of a subdivided wall
- layered wheels or variation of them

The End